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On the boundary dynamics of Chern-Simons gravity

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Abstract: We study Chern-Simons theory with a complex $G_{\mathbb{C}}$ or a real $G \times G$ gauge group on a manifold with boundary — this includes lorentzian and euclidean (anti-) de Sitter (E/A)dS gravity for $G = \mathrm{SU}(2)$ or $G = \mathrm{SL}(2, \mathbb{R})$. We show that there is a canonical choice of boundary conditions that leads to an unambiguous, fully covariant and gauge invariant, off-shell derivation of the boundary action — a $G_{\mathbb{C}}/G$ or G WZW model, coupled in a gauge invariant way to the boundary value of the gauge field. In particular, for (E/A)dS gravity, the boundary action is a WZW model with target space $(E/A)dS_3$, reminiscent of a worldsheet for worldsheet mechanism. We discuss in some detail the properties of the boundary theories that arise and we confront our results with various related constructions in the literature.

Keywords: Chern-Simons Theories, Models of Quantum Gravity, Conformal Field Models in String Theory.

Contents

1. Introduction and discussion

In this paper we revisit the issue of 'boundary dynamics' in three-dimensional gravity. The main goal is to uncover the universal nature of a natural set of boundary conditions and to give a completely general, covariant and off-shell derivation of the corresponding boundary action. In particular, we find that the target space of the boundary theory sigma model is equal to the model geometry of the bulk gravity theory, e.g. the boundary action of asymptotically de Sitter gravity is a Wess-Zumino-Witten model with target space three-dimensional de Sitter space dS_3, etc. Furthermore, we are able to give a somewhat unified discussion of other results in this field.

It is well known [1] that asymptotically anti-de Sitter (AdS) gravity has an infinite-dimensional algebra of asymptotic symmetries which is a Virasoro algebra with a non-trivial classical central charge (and likewise for de Sitter (dS) space [2, 3]). This fits in neatly with the AdS_3/CFT_2 correspondence based on string theory on AdS_3 and its holographically dual CFT — see e.g. [4, 5, 6].

On the other hand, three-dimensional gravity can be formulated [7, 8, 9] as a Chern-Simons gauge theory [10],[1] and this provides an (*a priori* different) relation between (A)dS gravity and CFT, manifested e.g. in the fact that Chern-Simons theory (quite generally)

[1] See [11, 12] for a derivation of the asymptotic Virasoro algebra in this context.

JHEP01(2003)067

induces a chiral Wess-Zumino-Witten (WZW) model on the boundary [13]. Beginning with the work of [14], attempts have been made to attribute a dynamical origin to this conformal symmetry purely within three-dimensional gravity, by relating this Virasoro algebra, along the lines of [13], to a boundary conformal field theory obtained by a suitable restriction of the (topological) bulk gravitational dynamics to the boundary. This 'boundary dynamics' is typically a Liouville theory obtained as a constrained WZW model from the on-shell value of the gravitational action, the constraints arising from the asymptotic boundary conditions.

This boundary Liouville action undoubtedly captures certain aspects of the gravitational dynamics — see e.g. [15, 16, 17, 18] for discussions in the gravity context and [19] for the emergence of Liouville theory in the so-called long string sector of string theory on AdS_3. But the overall significance of this boundary action for three-dimensional quantum gravity has remained somewhat unclear. In particular, there appears to be some controversy between the general relativity and string theory camps regarding the relation of the boundary action to a holographic description of three-dimensional gravity. For a clear discussion of these issues from a string theory perspective see [20] — for a different point of view see e.g. the discussion in [21].

We believe that this unsatisfactory state of affairs is in part due to the fact that the standard derivations of the boundary action, as described in [14] for AdS and in [22] for dS gravity, are rather non-covariant (requiring detailed assumptions about the asymptotics of the fields, the boundary, special choices of coordinates and/or gauge conditions), and make use of the bulk (and boundary) equations of motion. The latter, in particular, makes it difficult to assess the significance of the boundary action for three-dimensional quantum gravity.

In this note we provide a complementary approach to the the standard derivations of the boundary action. We will study Chern-Simons theory with a complex $G_{\mathbb{C}}$ [23] or a real $G \times G$ gauge group on an arbitrary three-manifold with boundary. This includes lorentzian and euclidean (anti-) de Sitter (E/A)dS gravity, with $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(1,1) \sim \mathrm{SL}(2,\mathbb{R})$ the 'Lorentz' group of local frame rotations. We will show that there is a canonical choice of boundary terms, fixed uniquely by the requirement that the complete bulk plus boundary action be G-invariant plus the requirement that half of the gauge fields are fixed on the boundary. In particular, in the gravitational context the former is just the requirement of invariance under local frame rotations, a natural requirement as this gauge symmetry is an artefact of the dreibein formulation of gravity, while the latter is compatible with the structure of asymptotically (E/A)dS gravity.

We will show that this choice of boundary terms leads to an unambiguous, fully covariant and gauge invariant, *off-shell* derivation of the boundary action, which turns out to be a $G_{\mathbb{C}}/G$ or G WZW model, coupled in a gauge invariant way to the boundary value of the gauge fields which act as sources for the currents of the WZW model. This coupling is such that, if these boundary gauge fields were to be treated as dynamical fields (in our approach they are not), then these actions would describe topological field theories, either the standard G/G models [24] or novel "$(G_{\mathbb{C}}/G)/G$" models, presumably related to the fusion rules of the $G_{\mathbb{C}}/G$ WZW model.

In particular, for (E/A)dS gravity, in each case the boundary action is a WZW model with target space the space(time) $(E/A)dS_3$ itself, i.e. the string theory sigma model with the NS target space B-field required by conformal invariance. E.g. for AdS_3 this is a WZW model with target space $\mathrm{SU}(1,1) \sim \mathrm{SL}(2,\mathbb{R})$, and this reproduces the result of [14], where the $\mathrm{SL}(2,\mathbb{R})$ WZW model appears at an intermediate step in the derivation of Liouville theory. But we believe that in their present generality the results are new, and that our general derivation provides a novel perspective and clarifies the role of the boundary action.

For example, we do not see any obvious decoupling of the bulk and boundary theories for quantities other than the bulk partition function, and our derivation shows very clearly that the complete bulk gravitational dynamics (including bulk correlation functions) is not captured by the boundary dynamics alone but by the combined bulk plus boundary action. We are thus not making any claims about a potentially dual two-dimensional field theory and about the holographic degrees of freedom.

Our result is also reminiscent of a worldsheet for worldsheet mechanism [25, 26]. To further develop and strengthen this possible analogy between the target space of our boundary action and the base space of our quantum gravity theory one would need to study the Wilson lines of Chern-Simons gravity and the corresponding boundary operator insertions.

We also want to emphasise that the present derivation circumvents what is perhaps the least attractive feature of the standard derivations — the necessity to combine two chiral WZW models into a non-chiral WZW model by using the boundary equations of motion.[2] In our derivation, this recombination is automatic and off-shell due to the Polyakov-Wiegman identities (the relevant Polyakov-Wiegman term being provided by the boundary term which ensures G-invariance of the action).

We close this Introduction with some comments on other related work.

Of all the articles on the boundary dynamics of three-dimensional gravity, [27] is perhaps the one closest in spirit to the present work. We will discuss the relation between [27] and our work in section 2.4. In [28] (see [29] for related work) the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ boundary WZW model is also interpreted as a sigma model with $EAdS_3$ target space without taking the Liouville limit. However, the derivation of the boundary action follows the procedure in [14], and thus suffers from the same shortcomings we discussed above. Moreover, our conclusions regarding the role of the boundary action are rather different. In [28], on the basis of the absence of local degrees of freedom in Chern-Simons theory, it is claimed that asymptotically EAdS gravity is equivalent to the dynamics of the boundary WZW model. As emphasised above, our present off-shell analysis strongly suggests that this is not the case at the quantum level. We agree with the analysis of Martinec [20] of these issues (but see [17, 21]).

The rest of the paper consists of section 2 in which we present the derivation and describe the significance of our boundary terms and section 3 wherein the properties of these actions, in particular of the coset WZW models, are described. In the appendix we present our conventions for three-dimensional gravity, Chern-Simons and WZW theories.

[2] In the case of dS gravity [22] this procedure is moreover ambiguous, leading to either an $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ or an $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(1,1)$ WZW model — clearly very different theories.

JHEP01(2003)067

2. Boundary terms for Chern-Simons gravity

2.1 Chern-Simons theory with gauge group $G_{\mathbb{C}}$ or $G \times G$

We will consider Chern-Simons theory with gauge group $G_{\mathbb{C}}$, the complexification of a (not necessarily compact) real Lie group G, or with gauge group $G \times G$, on a three-manifold M with boundary $\partial M = \Sigma$. We are interested in the odd (imaginary for $G_{\mathbb{C}}$) part

$$I_{CS}[A, \bar{A}] = I_{CS}[A] - I_{CS}[\bar{A}] \tag{2.1}$$

of the Chern-Simons action

$$I_{CS}[A] = \int_M \operatorname{tr}\left(A \wedge dA + \frac{2}{3} A^3\right), \tag{2.2}$$

where, for $G_{\mathbb{C}}$, A and $\bar{A}$ are $G_{\mathbb{C}}$-connections related by hermitean conjugation (see appendix A.1), and, for $G \times G$, independent G-connections.

It is well known [7, 8] that (modulo boundary terms, which will be the main focus of our discussion later on) this action for $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(1,1) \sim \mathrm{SL}(2,\mathbb{R})$ is (proportional to) the Palatini action for euclidean or lorentzian three-dimensional gravity with a positive or negative cosmological constant or, in other words, (euclidean) (anti-) de Sitter (E/A)dS gravity (see appendix A.3). This is summarized in the table below:

	$G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$	$G \times G$
$G = \mathrm{SU}(2)$	EAdS	EdS
$G = \mathrm{SU}(1,1)$	dS	AdS

(2.3)

In each case the model target space geometry $(E/A)dS_3$ can be realised as $G_{\mathbb{C}}/G$ or $(G \times G)/G \sim G$ respectively, e.g. $EAdS_3 \sim \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2) \sim H^3$ or $AdS_3 \sim (\mathrm{SU}(1,1) \times \mathrm{SU}(1,1))/\,\mathrm{SU}(1,1) \sim \mathrm{SU}(1,1)$, and G is the (euclidean or lorentzian) tangent space group.

The even (real) part $I_{CS}[A] + I_{CS}[\bar{A}]$ of the Chern-Simons action gives rise to the so-called 'exotic' action for three-dimensional gravity [7], involving a Chern-Simons action for the spin connection. Classically, for the vacuum Einstein equations, the ordinary Palatini action and this action are equivalent. But when following the usual minimal coupling prescription for matter, in the case of the exotic action the torsion is determined by the energy-momentum and the curvature by the spin-density, the opposite of what happens for the Palatini action. For this reason, we will not consider this action in the following.

2.2 G-invariance

We now turn to a discussion of boundary terms appropriate for describing the boundary dynamics of Chern-Simons gravity. As discussed in the Introduction, it is natural to require that even on a manifold with boundary the action be invariant under local Lorentz transformations (tangent space rotations) — these are an artefact of the vielbein formulation of gravity and bulk Lorentz invariance should not interfere with the boundary degrees of freedom. More generally this amounts to demanding strict invariance under gauge transformations taking values in $G \subset G_{\mathbb{C}}$ or in the diagonal subgroup $G \subset G \times G$.

$G_{\mathbb{C}}$ or $G \times G$ gauge transformations act on A and $\bar{A}$ as (see appendix A.1)

$$A \rightarrow A^{g} \equiv g^{-1} A g + g^{-1} d g \tag{2.4}$$

$$\bar{A} \rightarrow \bar{A}^{\bar{g}} \equiv \bar{g}^{-1} \bar{A} \bar{g} + \bar{g}^{-1} d \bar{g}\,, \tag{2.5}$$

with $g = \bar{g}$ iff g takes values in G. The Chern-Simons action transforms as

$$I_{CS}[A^{g}] = I_{CS}[A] - \frac{1}{3}\int_{M} \operatorname{tr}(g^{-1}dg)^{3} + \oint \operatorname{tr}(A \wedge dgg^{-1}) \tag{2.6}$$

($\oint = \int_{\Sigma}$) and hence under G-transformations ($g = \bar{g}$) the action $I_{CS}[A, \bar{A}]$ transforms as

$$I_{CS}[A^{g}, \bar{A}^{g}] = I_{CS}[A, \bar{A}] + \oint \operatorname{tr}(A - \bar{A}) \wedge dgg^{-1} \tag{2.7}$$

(we will analyse the behaviour under general gauge transformations in section 3.1). In particular, the WZ (winding number) term drops out.[3] It is clear that this variation can be cancelled by the gauge variation of the boundary term $\oint \operatorname{tr}(A \wedge \bar{A})$, because

$$\oint \operatorname{tr}(A^{g} \wedge \bar{A}^{g}) = \oint \operatorname{tr}(A \wedge \bar{A}) + \oint \operatorname{tr}(A - \bar{A}) \wedge dgg^{-1} + \oint \operatorname{tr}(g^{-1}dg \wedge g^{-1}dg) \tag{2.8}$$

and the last term is a total derivative and hence zero.

Thus the most general G-invariant gravitational action takes the form

$$I_{inv}[A, \bar{A}] = I_{CS}[A, \bar{A}] - \oint \operatorname{tr}(A \wedge \bar{A}) + \oint \mathcal{F}(A, \bar{A})\,, \tag{2.9}$$

where $\mathcal{F}(A, \bar{A})$ is a G-invariant functional of the boundary fields. Excluding higher-derivative terms, such a functional can only be algebraic and quadratic in $A - \bar{A}$.

We note in passing that the requirement of G-invariance can only be satisfied in the special case $k = 0$ of the general two-parameter family [23] of actions

$$(k + is) I_{CS}[A] + (k - is) I_{CS}[\bar{A}] \tag{2.10}$$

for $G_{\mathbb{C}}$ (and likewise for $G \times G$), precisely the case we have been considering. This is obvious from the above discussion because for a general linear combination of Chern-Simons actions the Wess-Zumino (or winding number) term in (2.6) will not disappear in the gauge variation, and this term cannot be cancelled by a *local* functional of the boundary gauge fields.

This also has an explanation in the language of geometric quantization. For $k \neq 0$ the prequantum line bundle is non-trivial, and hence gauge transformations are necessarily implemented on wave functions with a non-trivial cocycle. G-invariance of the action, on the other hand, means that the G-action is implemented trivially on the wave functions as $\psi(A) \rightarrow \psi(A^{g})$. Using the formalism of [23] it can indeed be checked that the prequantum operator determined by our choice of boundary terms reduces to the standard (trivial) generator of gauge transformations on functions of A for gauge transformations taking values in G.

[3] Therefore there is no quantization condition on the Chern-Simons coupling constant — which is why we have not worried about the normalization of the Chern-Simons action in (2.2).

2.3 Boundary conditions and boundary terms

It comes as a pleasant surprise that this choice of boundary terms, arising from the requirement of G-invariance, is compatible with the structure of asymptotically (EA)dS gravity where it is known that the natural boundary conditions are such that either A_z and $\bar{A}_{\bar{z}}$ or $A_{\bar{z}}$ and $\bar{A}_z$ are fixed on the euclidean boundary (and likwise for AdS). Indeed, as we will now show, these two requirements together uniquely determine the boundary terms.

The action appropriate for fixing either $(A_z, \bar{A}_{\bar{z}})$ or $(A_{\bar{z}}, \bar{A}_z)$ on the boundary is (cf. (A.18))

$$\begin{aligned} I^{\pm}[A,\bar{A}] &= I^{\pm}_{CS}[A] - I^{\mp}_{CS}[\bar{A}] \\ &= I_{CS}[A,\bar{A}] \pm \oint A_z A_{\bar{z}} \pm \oint \bar{A}_{\bar{z}} \bar{A}_z \,. \end{aligned} \tag{2.11}$$

Regardless of the boundary condition we denote the corresponding boundary value of the gauge field by $B = (B_z, B_{\bar{z}})$, so that for the two actions we have

$$I^+ : \;\; A_z|_{\partial M} = B_z \qquad \bar{A}_{\bar{z}}|_{\partial M} = B_{\bar{z}} \tag{2.12}$$

$$I^- : \;\; \bar{A}_z|_{\partial M} = B_z \qquad A_{\bar{z}}|_{\partial M} = B_{\bar{z}} \,. \tag{2.13}$$

Note that, for $G_{\mathbb{C}}$, the connection B, even though assembled from the $G_{\mathbb{C}}$ connections A and $\bar{A}$, is actually a G-connection, i.e. (see appendix A.1) it satisfies $B = \bar{B}$,

$$B = A_z dz + \bar{A}_{\bar{z}} d\bar{z} = \bar{B} \tag{2.14}$$

and likewise for $B = (\bar{A}_z, A_{\bar{z}})$.

The $G_{\mathbb{C}}$ gauge transformations on B inherited from the bulk gauge transformations acting on A and $\bar{A}$ is

$$I^+ : \;\; B = A_z dz + \bar{A}_{\bar{z}} d\bar{z} \to A^g_z dz + \bar{A}^{\bar{g}}_{\bar{z}} d\bar{z} \tag{2.15}$$

$$I^- : \;\; B = \bar{A}_z dz + A_{\bar{z}} d\bar{z} \to \bar{A}^{\bar{g}}_z dz + A^g_{\bar{z}} d\bar{z} \,, \tag{2.16}$$

e.g.

$$B = A_z dz - (A_z dz)^\dagger \to A^g_z dz - (A^g_z dz)^\dagger \,. \tag{2.17}$$

for $G = \mathrm{SU}(2)$ (see appendix A.1). We will write this as

$$B \to B^{(g,\bar{g})} \tag{2.18}$$

and need to keep in mind that only for g taking values in $G \subset G_{\mathbb{C}}$ this reduces to the standard gauge transformation $B \to B^g$ of a G gauge field,

$$B^{(g,g)} = B^g \,. \tag{2.19}$$

This is precisely the extension of the action of the gauge group $\mathcal{G} = \mathrm{Map}(\Sigma, G)$ to $\mathcal{G}_{\mathbb{C}}$ on the space $\mathcal{A}$ of G gauge fields (with the quotient $\mathcal{A}/\mathcal{G}_{\mathbb{C}}$ essentially the finite-dimensional moduli space of flat connections by the Narasimhan-Seshadri theorem) that one also encounters in the geometric quantization of $G_{\mathbb{C}}$ Chern-Simons theory [23] and that arises here very concretely and naturally.

JHEP01(2003)067

For the corresponding statements in the $G \times G$ case, see section 3.3.

To the action $I^{\pm}[A,\bar{A}]$ we are still free to add any (local) functional of the boundary gauge field B. It is straightforward to see that there is precisely one such term which makes the action G-invariant, i.e. compatible with the structure of $I_{inv}[A,\bar{A}]$ we determined above, namely $\mp 2\oint \operatorname{tr} B_z B_{\bar{z}}$.

Thus our desiderata have uniquely determined the action which we will consider to be

$$\begin{aligned} I^{\pm}_{\rm tot}[A,\bar{A}] &= I^{\pm}[A,\bar{A}] \mp 2\oint \operatorname{tr} B_z B_{\bar{z}} \\ &= I_{CS}[A,\bar{A}] - \oint \operatorname{tr}(A\wedge\bar{A}) \pm \oint \operatorname{tr}(A_z-\bar{A}_z)(A_{\bar{z}}-\bar{A}_{\bar{z}})\,. \end{aligned} \tag{2.20}$$

This shows that the two possibilities for $\mathcal{F}(A,\bar{A})$ compatible with the boundary conditions are $\mathcal{F}(A,\bar{A}) = \pm\oint \operatorname{tr}(A_z-\bar{A}_z)(A_{\bar{z}}-\bar{A}_{\bar{z}})$ respectively. The above action, as we have written it, is imaginary for $G_{\mathbb{C}}$. We find it more convenient to keep this in mind than to clutter subsequent equations with explicit factors of $i=\sqrt{-1}$.

As we will see, Chern-Simons theory with this choice of boundary terms has several attractive features. In particular, by construction, it is strictly G-invariant. We can thus anticipate the appearance of $G_{\mathbb{C}}/G$ or $(G\times G)/G$ cocycles when studying the behaviour of our gravitational action (or wave functions) under general gauge transformations.

2.4 Gravitational interpretation

The above construction works for any G, but it is instructive to look at the meaning of the boundary terms in the gravitational context or, for general G, in the 'gravitational parametrisation'

$$A = \omega + \eta e\,, \qquad \bar{A} = \omega - \eta e\,, \tag{2.21}$$

where ω is a G-connection and e is a one-form in the adjoint of G, which are, of course, to be identified with the spin-connection and dreibein in the gravitational context. Here $\eta = i$ for $G_{\mathbb{C}}$ (so that A and $\bar{A}$ are related by hermitean conjugation) and $\eta = 1$ for $G\times G$ (so that A and $\bar{A}$ are independent G-connections).

In terms of (ω, e), the Chern-Simons action is

$$I_{CS}[A,\bar{A}] = 4\eta\int \operatorname{tr}\left(e\wedge d\omega + e\wedge\omega\wedge\omega + \eta^2\frac{1}{3}e^3\right) + 2\eta\oint_{\Sigma} \operatorname{tr} e\wedge\omega\,. \tag{2.22}$$

Clearly the bulk part of the action (proportional to the Palatini action $I[e,\omega]$) is invariant under gauge transformations taking values in $G\subset G_{\mathbb{C}}$ or $G\subset G\times G$ since both e and the curvature $F(\omega) = d\omega + \omega\wedge\omega$ of ω transform homogeneously under G. The boundary term, however, is not.

One can recognize the terms that we have added to $I_{CS}[A,\bar{A}]$ as being proportional to the integral of the exterior curvature (essentially one half of the Gibbons-Hawking term),

$$\oint \operatorname{tr}(A\wedge\bar{A}) = 2\eta\oint \operatorname{tr}(e\wedge\omega) \tag{2.23}$$

and an area term,

$$\oint \mathrm{tr}(A_z - \bar{A}_z)(A_{\bar{z}} - \bar{A}_{\bar{z}}) = 4\eta^2 \oint \mathrm{tr}(e_z e_{\bar{z}})\,. \tag{2.24}$$

Note first of all that the sign of the exterior curvature term is such that it precisely cancels the exterior curvature boundary term in the action (2.22). Thus our action is just the sum of the Einstein-Palatini action and the area term. Since this action only depends on e and $F(\omega)$, both of which transform homogeneously under G, this makes it manifest that the action is strictly invariant under $G \subset G_{\mathbb{C}}$, i.e. under SU(2) or SU(1,1) local Lorentz transformations (frame rotations).

The sign of the area term is such that, for the appropriate choice of boundary condition, for $G = \mathrm{SU}(2)$ (EAdS) it *regularises* the action in the sense that e.g. the classical action for AdS_3 is zero. Thus the area term acts as a suitable counterterm [30, 31] to provide finite conserved quantities from the Brown-York [32] prescription (modulo logarithmic divergences). Likewise for dS [33].

Thus another way to arrive at the action we have been led to consider is to demand that it be invariant under Lorentz transformations (this rules out Gibbons-Hawking like terms) and finite on classical asymptotically (EA)dS solutions. This fixes the action uniquely up to Lorentz-invariant higher-derivative boundary terms (which do not contribute to the asymptotics in 3d [30]).

If we had added the two terms in (2.20) with the *opposite sign*, then the exterior curvature terms would have added up to the standard Gibbons-Hawking term (or, rather, its Palatini counterpart $I_{GH}[e,\omega]$), giving an action with a well-defined variational principle for the metric (or e) fixed on the boundary. The area term would still have regularized the classical action, since the Chern-Simons part of the action is finite all by itself, and thus all that is required is a cancellation of divergences between the area and exterior curvature terms. Thus we would have obtained the geometrodynamics action with e or the metric fixed on the boundary, with the appropriate counterterm $I_{ct}[e]$,

$$I_{CS}[A]-I_{CS}[\bar{A}]+\oint \mathrm{tr}(A\wedge\bar{A})\mp\oint \mathrm{tr}(A_z-\bar{A}_z)(A_{\bar{z}}-\bar{A}_{\bar{z}}) \sim I[e,\omega]+I_{GH}[e,\omega]+I_{ct}[e]\,. \tag{2.25}$$

This is precisely the action Krasnov was led to investigate in the Chern-Simons formulation of EAdS gravity in [27]. However, for reasons we will now explain we believe that the action he actually ends up studying is the action (2.20) instead.[4]

One intriguing aspect of our choice of action, which will appear as a byproduct of our general analysis in the next section, but which can also readily be seen directly by using the Polyakov-Wiegman (PW) identities (A.25), is that on pure gauge configurations, $A = g^{-1}dg, \bar{A} = \bar{g}^{-1}d\bar{g}$, it reduces to a WZW model (A.24) with argument $g\bar{g}^{-1}$ (3.4). This reflects the strict invariance of the action under G-transformations, for which $g = \bar{g}$.

The action (2.25) considered by Krasnov does not have this invariance, as it contains the (non-invariant) Gibbons-Hawking term, and indeed it can be checked that on-shell (on pure gauge transformations) the classical action does not combine nicely into a single

[4]The author of [27] agrees with this conclusion [34].

WZW model courtesy of the PW identities. But precisely such a combination is claimed to occur in [27], and this is a property of (2.20), not of the geometrodynamics action (2.25).

3. The boundary action for Chern-Simons gravity

We find it notationally more convenient to treat the two cases $G_{\mathbb{C}}$ and $G \times G$ separately, even though (as will be seen) they are exactly analogous. We begin with $G_{\mathbb{C}}$, deferring a discussion of $G \times G$ to section 3.3.

3.1 From Chern-Simons gravity to $G_{\mathbb{C}}/G$ WZW models

To establish the emergence of $G_{\mathbb{C}}/G$ models from Chern-Simons gravity, anticipated in various ways above, we now determine the behaviour of $I^{\pm}_{\mathrm{tot}}[A, \bar{A}]$ under a general $G_{\mathbb{C}}$ gauge transformation.

First of all, note that gauge variation of the Chern-Simons part $I^{\pm}[A, \bar{A}]$ of the total action is the sum of a chiral and an anti-chiral WZW action (A.23),

$$\begin{aligned} I^{+}[A^{g}, \bar{A}^{\bar{g}}] &= I^{+}[A, \bar{A}] + (I^{+}[g, A_{z}] + I^{-}[\bar{g}, \bar{A}_{\bar{z}}]) \\ I^{-}[A^{g}, \bar{A}^{\bar{g}}] &= I^{-}[A, \bar{A}] - (I^{-}[g, A_{\bar{z}}] + I^{+}[\bar{g}, \bar{A}_{z}])\,. \end{aligned} \tag{3.1}$$

Usually at this stage [14, 22] (after having gone 'on-shell' in the bulk) it is then argued that (on-shell on the boundary) the sum of these two actions is equivalent to a non-chiral WZW action. In the present case, however, this is much clearer and completely unambiguous since the boundary terms we have added to enforce G-invariance precisely provide the relevant Polyakov-Wiegman terms that allow us to combine these two chiral actions *off-shell* into a single non-chiral action for the composite field $g\bar{g}^{-1}$.

Indeed a reasonably straightforward calculation, using the PW identities (A.25) shows that

$$I^{\pm}_{\mathrm{tot}}[A^{g}, \bar{A}^{\bar{g}}] = I^{\pm}_{\mathrm{tot}}[A, \bar{A}] \pm I^{\pm}[g\bar{g}^{-1}, B] \tag{3.2}$$

where

$$\begin{aligned} I^{+}[h, B] &= I^{+}[h] + 2\oint \mathrm{tr}(B_{z}\partial_{\bar{z}}hh^{-1} - B_{\bar{z}}h^{-1}\partial_{z}h) + 2\oint \mathrm{tr}(B_{z}B_{\bar{z}} - B_{z}hB_{\bar{z}}h^{-1}) \\ I^{-}[h, B] &= I^{-}[h] - 2\oint \mathrm{tr}(B_{z}h^{-1}\partial_{\bar{z}}h - B_{\bar{z}}\partial_{z}hh^{-1}) + 2\oint \mathrm{tr}(B_{z}B_{\bar{z}} - B_{z}h^{-1}B_{\bar{z}}h) \\ &= I^{+}[h^{-1}, B] \end{aligned} \tag{3.3}$$

and where $I^{\pm}[h]$ is the WZW action (A.24). We will discuss these actions in more detail in the next section.

Note that the above result is completely general:

- No assumptions have been made about the topology of M or Σ, or about $G_{\mathbb{C}}$. In particular, the results hold for Σ the boundary of any region M in space(-time), not only an 'asymptotic' boundary.

JHEP01(2003)067

- We did not need to make use of the bulk equations of motion to arrive at the boundary action. In particular, no special coordinatization was required to turn some of the components of the gauge field into Lagrange multipliers enforcing the bulk flatness conditions.
- We did not need to go on-shell for the boundary theory to combine two chiral WZW models into a standard WZW model.

In all these aspects our derivation of the boundary action differs from the standard derivations. We believe that our approach displays more clearly the role of the boundary action, in particular in the full bulk quantum gravity theory.

3.2 Properties of the boundary action

We list here some basic properties of the boundary actions (cocycles) $I^{\pm}[h, B]$.

1. With dynamical boundary gauge fields B, the actions $I^{\pm}[h, B]$ with $h \in G_{\mathbb{C}}$ are the actions of diagonally gauged (thus anomaly free) $G_{\mathbb{C}}/G$ WZW models.

   The action we find is such that h is restricted to take values in $G_{\mathbb{C}}/G$, $h = g\bar{g}^{-1}$. Thus these are, despite appearance, somewhat unusual WZW models.

   In particular, since the cocycle $I^{\pm}[g\bar{g}^{-1}, B]$ depends on g and $\bar{g}$ only in the combination $g\bar{g}^{-1} \in G_{\mathbb{C}}/G$, it follows immediately that the gravitational action I_{tot} is invariant under gauge transformations with $g = \bar{g}$, i.e. under local frame rotations, as expected, because $I^{\pm}[h = 1, B] = 0$,

   Note, incidentally, that these "$(G_{\mathbb{C}}/G)/G$" models can be shown to be topological field theories, non-compact analogues of the more familiar G/G models [24], which are of interest in their own right — these will be studied elsewhere.

2. In our case, however, the B are non-dynamical and these theories are best thought of as WZW models coupled in a gauge-invariant way to external currents B. Setting these currents to zero, or evaluating the gravitational action on classical solutions which are pure gauge, $A = 0^g = g^{-1}dg$, $\bar{A} = \bar{0}^{\bar{g}}$, one obtains the action

$$I^{\pm}_{\text{tot}}[0^g, \bar{0}^{\bar{g}}] = \pm I^{\pm}[g\bar{g}^{-1}, B = 0] = \pm I^{\pm}[g\bar{g}^{-1}]\,. \tag{3.4}$$

   These WZW models are sigma models with target space $G_{\mathbb{C}}/G$ and a non-trivial (string theory) B-field B_{NS}.

3. These actions (and their gauged counterparts) are (modulo an overall factor of i) *real*, in contrast to ordinary (gauged) WZW models in euclidean signature which have a complex action. (For example, the WZ term is real for unitary matrices but imaginary for (hermitean) matrices of the form $g\bar{g}^{-1}$). This is of course a consequence of the fact that they arise as cocycles of the real gravitational action we have been considering. Also, for G compact they have a positive (semi-)definite kinetic term and are thus well defined in euclidean signature. Their continuation to lorentzian signature, however, would be *complex*, hence these theories are presumably non-unitary in Minkowski space.

4. For $G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$ the target space of the WZW model is the three-dimensional space(-time) $(EA)dS_3$, precisely the three-manifold whose quantum gravity we set out to describe. Thus, for any three-manifold M, the boundary action is a sigma model with target space $(EA)dS_3$, corresponding to the 'on-shell' base space M of the Chern-Simons gravity theory. We will see that the same thing happens for $G \times G$.

   This is reminiscent of the worldsheet for worldsheet construction [25] in $N = 2$ string theory [26]. In that case one finds that the beta-function equations (that is the conditions for that theory to be conformally invariant — quantum consistent) of the worldsheet conformal field theory give rise to a target space action, the field content and symmetry of which are identical to the original worldsheet theory. In our case we find that the symmetries of our three-dimensional base space theory give rise to a target space theory of the same structure.

   (*a*) In particular, for $G = \mathrm{SU}(2)$, with the parametrisation

$$g\bar{g}^{-1} = \begin{pmatrix} e^{\phi}u\bar{u} + e^{-\phi} & e^{\phi}u \\ e^{\phi}\bar{u} & e^{\phi} \end{pmatrix} \tag{3.5}$$

$$g = \begin{pmatrix} e^{-\phi/2} & e^{\phi/2}u \\ 0 & e^{\phi/2} \end{pmatrix} \tag{3.6}$$

   of the coset $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$, the action of the WZW model is

$$I^{+}[g\bar{g}^{-1}] = 2\int (\partial\phi\bar{\partial}\phi + e^{2\phi}\partial u\bar{\partial}\bar{u})\,. \tag{3.7}$$

   This action has been studied in detail in the past e.g. in [35, 36]. It is also part of the string theory sigma model on $EAdS_3$ with non-zero NS B-field

$$ds^2 = d\phi^2 + e^{2\phi}dud\bar{u}\,, \qquad B_{NS} = e^{2\phi}du \wedge d\bar{u} \tag{3.8}$$

   as studied in detail e.g. in [4, 5, 6]. In [36] one sees clearly that the correlation functions are closely related in construction and form to those of Liouville theory. And in the limit that the string worldsheets are localized at radial infinity, the worldsheet action reduces to Liouville theory [19].

   (*b*) Likewise, for $G = \mathrm{SU}(1,1)$ (this amounts to replacing $(u,\bar{u}) \to (iu, i\bar{u})$ in the above) we find a target space with the dS_3 metric and an NS B-field,

$$ds^2 = d\phi^2 - e^{2\phi}dud\bar{u}\,, \qquad B_{NS} = -e^{2\phi}du \wedge d\bar{u}\,. \tag{3.9}$$

   This action can be reduced to Liouville theory by imposing further constraints appropriate for asymptotically de Sitter gravity [22, 37]. However, not much appears to be known about the properties of this sigma model itself. An initial step in the study of this model could be a minisuperspace analysis as carried out for the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ model in [38].

5. In the path integral, the cocycle gives rise to wavefunction(al)s

$$\Psi^{\pm}[B] = \int D[A]D[\bar{A}]\mathrm{e}^{isI^{\pm}_{\mathrm{tot}}[A,\bar{A}]}\,, \tag{3.10}$$

(s is real for euclidean AdS_3 and imaginary for lorentzian dS_3 [23]) of the boundary gauge field B which transform with a cocycle under $G_{\mathbb{C}}$ gauge transformations,

$$\Psi^{\pm}[B^{(g,\bar{g})}] = \mathrm{e}^{\pm isI^{\pm}[g\bar{g}^{-1},B]}\Psi^{\pm}[B]\,. \tag{3.11}$$

6. To understand the role of the $G_{\mathbb{C}}/G$ sigma model (cocycle) in the quantization of Chern-Simons gravity, one can gauge fix the above path integral by following the usual Faddeev-Popov procedure. Because of the non-trivial cocycle, the integral over the gauge group will not factor out of the path integral. Rather, one finds (see e.g. [39]) that the gauge fixed version of the wave function factorizes into the standard gauge fixed Chern-Simons gravity path integral $Z_{CS}[B]$ and the partition function of the WZW action $I^{\pm}[g^{-1}\bar{g},B]$, with the B treated as external sources,

$$\Psi[B] = Z_{CS}[B] \times Z_{G_{\mathbb{C}}/G}[J=B]\,. \tag{3.12}$$

In particular, the latter is a generating functional for correlation functions of the left and right h-currents in the WZW model (3.4) with target space $G_{\mathbb{C}}/G$. The addition of Wilson lines, gravitational sources, ending on the boundary will couple the bulk and boundary theories in a non-trivial way and will lead to operator insertions in the boundary partition function.

3.3 An analogous construction for $G \times G$

As we noted before, if the gauge fields A and $\bar{A}$ are taken to be *independent* G gauge fields rather than complex conjugate $G_{\mathbb{C}}$ gauge fields, then the (real) action $I_{CS}[A] - I_{CS}[\bar{A}]$ describes lorentzian anti-de Sitter (ADS) gravity for $G = \mathrm{SU}(1,1)$ or $G = \mathrm{SL}(2,\mathbb{R})$, and euclidean de Sitter (EdS) gravity for $G = \mathrm{SU}(2)$.

Most of what we have done above for $G_{\mathbb{C}}$ Chern-Simons gravity goes through verbatim in these cases as well with the replacements

$$G_{\mathbb{C}} \to G \times G \qquad G \to \mathrm{diag}(G \times G)\,. \tag{3.13}$$

As EdS and AdS have some slightly different features, we treat them separately, beginning with AdS.

For AdS we might be interested in a lorentzian boundary, and hence the boundary gauge field would be

$$B = A_{\pm}dx^{\pm} + \bar{A}_{\mp}dx^{\mp}\,. \tag{3.14}$$

This is an $\mathrm{SU}(1,1)$ or $\mathrm{SL}(2,\mathbb{R})$ gauge field, $B^{\dagger} = -\sigma_3 B \sigma_3$, and transforms as such under the diagonal subgroup of $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$. The two components of B transform independently under $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$, and this extension of the $\mathcal{G}$-action on G gauge fields to a $\mathcal{G} \times \mathcal{G}$ action is the analogue in the present case of the extension of the $\mathcal{G}$-action to $\mathcal{G}_{\mathbb{C}}$ encountered before.

In this case g and $\bar{g}$ are independent elements of SU(1,1), and $g\bar{g}^{-1}$ is thus an element of $(G \times G)/G \sim G$ itself. The cocycle we obtain in this case is the standard $G = \mathrm{SU}(1,1)$ or $G = \mathrm{SL}(2, \mathbb{R})$ WZW model which is real in lorentzian signature. Note that once again the boundary action is thus a sigma model with target space the three-manifold $G \sim AdS_3$ whose quantum gravity we set out to describe. By imposing further asymptotically AdS_3 constraints, this model can be reduced to Liouville theory [14]. The coupling to the boundary gauge fields B is such that with the latter treated as dynamical the action is actually precisely that of the *topological* G/G model.

For EdS, on the other hand, the boundary gauge field

$$B = A_z dz + \bar{A}_{\bar{z}} d\bar{z} \tag{3.15}$$

(say) is *not* an SU(2) gauge field, $B^\dagger \neq -B$, because A and $\bar{A}$ are independent connections (we would have encountered a similar situation for dS gravity with a lorentzian boundary). Nevertheless, under the diagonal subgroup of $G \times G$ it transforms as such and this is all that we will need.

Consequently we are now choosing complex boundary conditions on the two independent fields A_z and $\bar{A}_{\bar{z}}$. The gravitational action we are led to with these boundary conditions and the requirement of G-invariance is the sum of the standard Palatini action and an *imaginary* area term. While this may appear strange from the gravitational point of view, this is what emerges from our construction. However, these boundary conditions are most likely not natural for EdS.

The cocycle is now an SU(2) WZW model (in agreement with the fact that $EdS_3 \sim S^3 \sim \mathrm{SU}(2)$), which is complex in euclidean space (and unitary in lorentzian signature). With the B's included as dynamical fields, it is a topological SU(2)/SU(2) model (with slightly unusual reality properties).

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A. Conventions

A.1 Lie algebras, gauge fields and gauge transformations

Given a basis τ_a of the Lie algebra $\mathfrak{g}$ of G, $\{\tau_a, i\tau_a\}$ are a basis of $\mathfrak{g}_{\mathbb{C}}$. A $G_{\mathbb{C}}$ connection A can then be written as

$$A = \omega + ie = \tau_a(\omega^a + ie^a)\,. \tag{A.1}$$

We also define the conjugate connection $\bar{A}$ by [23]

$$\bar{A} = \omega - ie = \tau_a(\omega^a - ie^a)\,. \tag{A.2}$$

Thus $\bar{A} = -A^\dagger$ if the τ_a are anti-hermitean, $\tau_a^\dagger = -\tau_a$, but not in general. In particular, $\bar{A} = A$ iff A is a G-connection.

Likewise, for $G \times G$, we parametrise the two independent G-connections A and $\bar{A}$ as

$$\begin{aligned} A &= \omega + e = \tau_a(\omega^a + e^a) \\ \bar{A} &= \omega - e = \tau_a(\omega^a - e^a)\,, \end{aligned} \tag{A.3}$$

which satisfy $A^\dagger = -A$ and $\bar{A}^\dagger = -\bar{A}$ if the τ_a are anti-hermitean.

For $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(1,1)$ we have $G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) \sim \mathrm{SO}(3,1)$ and $G \times G \sim \mathrm{SO}(4)$ or $\sim \mathrm{SO}(2,2)$. For $G = \mathrm{SU}(2)$ we choose $\tau_a = -i\sigma_a/2$, $a = 1,2,3$ (σ_a are standard hermitean Pauli matrices), and thus $A^\dagger = -\bar{A}$, while for $G = \mathrm{SU}(1,1)$ we take $\tau_a = (-i\sigma_3/2, \sigma_1/2, \sigma_2/2)$, $a = 0,1,2$ with $A^\dagger = -\sigma_3 \bar{A} \sigma_3$. The τ_a satisfy

$$[\tau_a, \tau_b] = \epsilon_{abc}\tau^c \tag{A.4}$$

where indices are raised and lowered with δ_{ab} or $\eta_{ab} = \mathrm{diag}(-++)$ and we use the convention that $\epsilon_{123} = \epsilon_{012} = 1$. Explicitly, the curvature $F_A = dA + A^2$ of A is

$$\begin{aligned} F_A = \tau_a &\left(d\omega^a + \frac{1}{2}\epsilon^a_{\ bc}\omega^b \wedge \omega^c - \frac{1}{2}\epsilon^a_{\ bc} e^b \wedge e^c \right) + \\ &+ i\tau_a (de^a + \epsilon^a_{\ bc}\omega^b \wedge e^c) \end{aligned} \tag{A.5}$$

for $G_{\mathbb{C}}$, with an analogous expression for $G \times G$ obtained by sending $e \to -ie$.

$G_{\mathbb{C}}$ gauge transformations act on A and $\bar{A}$ as

$$A \to A^g \equiv g^{-1}Ag + g^{-1}dg \tag{A.6}$$
$$\bar{A} \to \bar{A}^{\bar{g}} \equiv \bar{g}^{-1}\bar{A}\bar{g} + \bar{g}^{-1}d\bar{g}\,. \tag{A.7}$$

with $\bar{g}^{-1} = g^\dagger$ for $G = \mathrm{SU}(2)$, and $\bar{g}^{-1} = \sigma_3 g^\dagger \sigma_3$ for $G = \mathrm{SU}(1,1)$ respectively.

Note that in both cases $g \to \bar{g}$ is an (outer) automorphism of $G_{\mathbb{C}}$, $\overline{gh} = \bar{g}\bar{h}$, which fixes $G \subset G_{\mathbb{C}}$, i.e. $g = \bar{g} \Leftrightarrow g \in G$. Thus $g \to g\bar{g}^{-1}$ is the projection from $G_{\mathbb{C}}$ to the coset $G_{\mathbb{C}}/G$. In particular, for $G = \mathrm{SU}(2)$ this is the projection $g \to h = gg^\dagger$ onto (positive) hermitean matrices, $h^\dagger = h$.

Likewise, for the $G \times G$ theory, g and $\bar{g}$ are independent G gauge transformations, and the map $g \to g\bar{g}^{-1}$ is the projection from $G \times G$ to G with kernel the diagonal subgroup $G_d = \{(g,g) \in G \times G\}$.

A.2 Chern-Simons and WZW actions

The Chern-Simons action is

$$I_{CS}[A] = \int \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A^3 \right). \tag{A.8}$$

Under a gauge transformation

$$A \to A^g \equiv g^{-1}Ag + g^{-1}dg \tag{A.9}$$
$$\bar{A} \to \bar{A}^{\bar{g}} \equiv \bar{g}^{-1}\bar{A}\bar{g} + \bar{g}^{-1}d\bar{g}\,. \tag{A.10}$$

the Chern-Simons action transforms as

$$I_{CS}[A^g] = I_{CS}[A] + C[g, A] \tag{A.11}$$

where the cocycle $C[g, A]$ is

$$C[g, A] = -\frac{1}{3}\int_M \mathrm{tr}(g^{-1}dg)^3 + \oint \mathrm{tr}(A \wedge dgg^{-1}) \tag{A.12}$$

and analogously for $I_{CS}[\bar{A}]$.

The variation of $I_{CS}[A]$ is

$$\delta I_{CS}[A] = 2\int \mathrm{tr}\, \delta A \wedge F_A - \oint \mathrm{tr}\, A \wedge \delta A\,. \tag{A.13}$$

Choosing a complex structure J on the boundary, in terms of complex coordinates the boundary term reads

$$\oint \mathrm{tr}\, A \wedge \delta A = \oint \mathrm{tr}(A_z \delta A_{\bar{z}} - A_{\bar{z}} \delta A_z) \tag{A.14}$$

where we introduce the convention that any expression of the form $\oint \mathcal{O}_{z\bar{z}}$ is to be interpreted as

$$\oint \mathcal{O}_{z\bar{z}} \equiv \oint dz \wedge d\bar{z}\; \mathcal{O}_{z\bar{z}}\,. \tag{A.15}$$

This makes it easy to switch between differential form and complex notation,

$$\oint A \wedge B = \oint (A_z B_{\bar{z}} - A_{\bar{z}} B_z)\,, \tag{A.16}$$

but has the drawback that terms that look real are imaginary, and vice-versa, because $dz \wedge d\bar{z}$ is imaginary,

$$\overline{dz \wedge d\bar{z}} = -dz \wedge d\bar{z}\,. \tag{A.17}$$

To obtain an action that is suitable for fixing either A_z or $A_{\bar{z}}$ on the boundary, we thus introduce the actions

$$I^{\pm}_{CS}[A] = I_{CS}[A] \pm \oint \mathrm{tr}\, A_z A_{\bar{z}} \tag{A.18}$$

which have the on-shell variations

$$\begin{aligned} \delta I^{+}_{CS}[A] &= 2\oint \mathrm{tr}\, A_{\bar{z}} \delta A_z \\ \delta I^{-}_{CS}[A] &= -2\oint \mathrm{tr}\, A_z \delta A_{\bar{z}}\,. \end{aligned} \tag{A.19}$$

We can thus fix the boundary conditions

$$A_z|_{\partial M} = B_z \qquad \text{or} \qquad A_{\bar{z}}|_{\partial M} = B_{\bar{z}} \tag{A.20}$$

for $I^{+}_{CS}[A]$ or $I^{-}_{CS}[A]$ respectively.

JHEP01(2003)067

One can proceed analogously if one wishes to consider a boundary with a lorentzian metric, as for asymptotically AdS space-times. In this case, one replaces the $(z, \bar{z})$ by light-cone coordinates $x^{\pm}$, with $dx^{+} \wedge dx^{-}$ real.

Under a gauge transformation, one finds that $I^{\pm}_{CS}[A]$ transforms as

$$I^{+}_{CS}[A^g] = I^{+}_{CS}[A] + I^{+}[g, B_z] \tag{A.21}$$

$$I^{-}_{CS}[A^g] = I^{-}_{CS}[A] - I^{-}[g, B_{\bar{z}}]\,, \tag{A.22}$$

where

$$\begin{aligned} I^{+}[g, B_z] &= I^{+}[g] + 2\oint \operatorname{tr} B_z \partial_{\bar{z}} g g^{-1} \\ I^{-}[g, B_{\bar{z}}] &= I^{-}[g] + 2\oint \operatorname{tr} B_{\bar{z}} \partial_{z} g g^{-1} \end{aligned} \tag{A.23}$$

with $I^{\pm}[g]$ the WZW actions

$$I^{\pm}[g] \equiv \oint \operatorname{tr}(g^{-1}\partial_z g g^{-1}\partial_{\bar{z}} g) \mp \frac{1}{3}\int_M \operatorname{tr}(g^{-1}dg)^3\,. \tag{A.24}$$

They satisfy the Polyakov-Wiegman (PW) identities

$$\begin{aligned} I^{+}[gh] &= I^{+}[g] + I^{+}[h] + 2\oint \operatorname{tr} g^{-1}\partial_z g \partial_{\bar{z}} h h^{-1} \\ I^{-}[gh] &= I^{-}[g] + I^{-}[h] + 2\oint \operatorname{tr} g^{-1}\partial_{\bar{z}} g \partial_{z} h h^{-1}\,. \end{aligned} \tag{A.25}$$

To make contact with the usual way of writing the WZW action, note that with $d^2 z = |dz \wedge d\bar{z}|$ we have

$$-iI^{\pm}[g] = -\oint d^2 z \operatorname{tr}(g^{-1}\partial_z g g^{-1}\partial_{\bar{z}} g) \pm \frac{i}{3}\int_M \operatorname{tr}(g^{-1}dg)^3\,. \tag{A.26}$$

A.3 SL$(2,\mathbb{C})$ Chern-Simons gravity

We review the Chern-Simons formulation of EAdS and dS gravity. We omit the discussion for AdS and EdS gravity which is precisely analogous.

In the first order formalism the fields are a dreibein (orthonormal frame) e^a and the spin connection $\omega^a_{\ b}$. We use the convention that $a = 1, 2, 3$ for $EAdS_3$ and $a = 0, 1, 2$ for dS_3.

The torsion-free condition for the spin-connection is

$$de^a + \omega^a_{\ b} \wedge e^b = 0\,. \tag{A.27}$$

The curvature two-form is

$$R^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b}\,. \tag{A.28}$$

In three dimensions, the Einstein equations with negative (positive) cosmological constant are equivalent to the constant curvature condition

$$R^a_{\ b} = \mp e^a \wedge e_b\,, \tag{A.29}$$

where indices are raised and lowered with the orthonormal (euclidean or minkowskian) tangent space metric δ_{ab} or $\eta_{ab} = \mathrm{diag}(- + +)$. Introducing the curvature radius by scaling $e^a \to e^a/\ell$, the cosmological constant could be shifted away from its canonical value $\Lambda = \pm 1$.

To establish the connection of these equations with the equations of motion of Chern-Simons theory, one introduces the dual spin connection

$$\omega^a = \mp\frac{1}{2}\epsilon^{abc}\omega_{bc} \tag{A.30}$$

where $\epsilon_{123} = 1$, $\epsilon_{012} = 1$ and the upper (lower) sign refers to $EAdS_3$ (dS_3) respectively. Then the equations become

$$\begin{aligned} de^a + \epsilon^a_{\ bc}\omega^b \wedge e^c &= 0 \\ d\omega^a + \frac{1}{2}\epsilon^a_{\ bc}\omega^b \wedge \omega^c &= \frac{1}{2}\epsilon^a_{\ bc}e^b \wedge e^c\,. \end{aligned} \tag{A.31}$$

Comparing with (A.5), we see that these are precisely the equations $F_A = 0$ of the $G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$-connection A for $G = \mathrm{SU}(2)$ or $G = \mathrm{SU}(1,1)$.

The relation between the Chern-Simons action and the Palatini action for three-dimensional (EA)dS gravity,

$$I[e,\omega] = \int e^a \wedge (d\omega + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c) - \frac{1}{6}\epsilon_{abc}e^a \wedge e^b \wedge e^c\,, \tag{A.32}$$

is

$$I_{CS}[A] - I_{CS}[\bar{A}] = \mp 2i(I[e,\omega] \mp \frac{1}{2}\oint e^a \wedge \omega) \tag{A.33}$$

(the signs arise beause for $G = \mathrm{SU}(2)$ we are raising and lowering indices with δ_{ab}, not with $\mathrm{tr}\,\tau_a\tau_b \sim -\delta_{ab}$). This action thus descibes euclidean gravity with a negative cosmological constant for $G = \mathrm{SU}(2)$, and lorentzian gravity with a positive cosmological constant for $G = \mathrm{SU}(1,1)$.

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JHEP01(2003)067

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JHEP01(2003)067

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